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Iterative Induced Sequence on Cone Metric Spaces using Fixed Point Theorems of Generalized Lipschitzian Map

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Abstract

In this paper, the concept of Lipschitzian map on cone metric spaces was examined. With this modification, some fixed point theorems of generalized Lipschitz mappings with weaker conditions on generalized Lipschitz constants were proved. Let (X,d) be a complete metric space and $T:X \to X$ a generalized Lipschitzian map with constant $k<1$. First, defining the sequence $\{x_n\}_{n\geq 1}$ inductively by $x_{n+1}=Tx_n$, $n=1,2,3,...,$ $x_0 \in X$ and if x^* is the unique fixed point of T, it was shown that $x_n \to x^*$ as $n \to \infty$ and $d(x_n, x^*) \leq \frac{k^n}{1-k} d(x_1, x_0)$.

Moreover, we shall assume that if X is complete cone metric space, P be a normal cone with normal constant K, T satisfying the generalized Lipschtizian condition where $0 \leq k \leq 1$ and for any x in X, then we will prove that the iterative induced defined sequence $\{Tx_n\}_{n\geq 1}$ converges to the fixed point.

Keywords: Cone Metric Spaces, Geneneralized Lipschtizian Condition, Fixed Point Theorems, Sequences.

Introduction

In many areas of applied mathematics, the existence of a fixed point of a certain function between metric spaces has a remarkable significance since it is equivalent to the existence of a solution in numerous mathematical problems.

Cone metric spaces were introduced by Huang and Zhang (2007) as a generalization of metric spaces. The distance $d(x,y)$ of two elements x and y in a cone metric space X is defined to be a vector in an ordered Banach space E, and a mapping $T:X \to X$ is said to be contractive if there is a constant $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. The right-hand side of inequality is the vector as the result of the operation of scalar multiplication in cone metric spaces.

If the Lipschitz constant is less than 1, then T is said to be a contraction.

The problem to solve is to prove that the iterative induced defined sequence $\{Tx_n\}_{n\geq 1}$ converges to the fixed point.

- **Definition 1.1.** Following Liu and Xu (2013): Let E always be a real Banach space such that for all $x, y, z \in E, \alpha \in R$:

1. $(xy)z = x(yz)$
2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$

4 $\| xy \| \leq \| x \| . \| y \|$

- **Definition 1.2.** Following Ili¢ and Rakocevi¢ (2009): Let E always be a real Banach space and P a subset of E, then P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{0\}$

(ii) $\{0,1\} \subset P$

(iii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$

(iv) $x \in P, -x \in P \Rightarrow x = 0$.

- **Definition 1.3.** Following Altun *et al.* (2010): Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of P.
- **Definition 1.4.** Following Huang and Zhang (2007): The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is sequence such that $x_1 \leq x_2 \leq \ldots \leq x_n \ldots \leq y$ for some $y \in E$, then there is $x^* \in E$ such that $\| x_n - x^* \| \to 0$ (as $n \to \infty$). Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

  In the following, we always suppose E is a Banach space, P is a cone in E with $\operatorname{int} P \neq \phi$ and $\leq$ is partial ordering with respect to P.

- **Definition 1.5.** Following Kumar and Ansari (2017): Let (X, d) be a cone metric space. Let $\{x_n\}_{n\geq 1}$ be a sequence in X and $x^* \in X$. The sequence $\{x_n\}_{n\geq 1}$ is said to be convergent to a point $x^* \in X$ if and if for all $c \in E$ with $0 << c$, there is N such that for all $n > N$, we have that $d(x_n, x^*) << c$. We denote this by

$\lim_{n\to\infty} x_n = x^*$ or $x_n \to x^*$ as $n \to \infty$.

- **Definition 1.6.** Following Huang and Zhang (2007): Let (X,d) be a cone metric space, $\{x_n\}_{n\geq 1}$ be a sequence in X. The sequence $\{x_n\}_{n\geq 1}$ is called a Cauchy sequence if and only if for all $c \in E$ with $0 << c$, there is N such that for all $n,m > N$, we have that $d(x_n, x_m) << c$.
- **Definition 1.7.** Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.
- **Definition 1.8.** Let (X,d) be a cone metric space. If for any sequence $\{x_n\}_{n\geq 1}$ in X, there is a subsequence $\{x_{n_i}\}_{i\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $\{x_{n_i}\}_{i\geq 1}$ is convergent in X. Then X is called a sequentially compact cone metric space.
- **Definition 1.9.** The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|$.

The least positive number satisfying above is called the normal constant of P. It is well known that a regular cone is a normal cone.

1. SOME CLASSES OF MAPPING

- **Definition 2.1.** Following Chidume (2003): Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies

(d1) $d(x,y) > 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$

(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$

(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space. It is obvious that cone metric spaces generalize metric spaces.

Example 1. Let $E = R^2, P = \{(x,y) \in E \mid x, y \geq 0\} \subset R^2, X = R$ and $d: X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha |x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Example 2. Let $E = R^n$ with $P = \{(x_1, x_2, x_3, ..., x_n) \in R^n : x_i \geq 0$ for all $i = 1,2,3,...,n\}$.

Let $X = R$ and $d: X \times X \to E$ such that

$d(x,y) = (|x-y|, \alpha_1 |x-y|, \alpha_2 |x-y|, \alpha_3 |x-y|, .. \alpha_{n-1} |x-y|)$ where $\alpha_i \geq 0$ for all $1 \leq i \leq n-1$. Then (X,d) is a cone metric space.

- **Definition 2.2.** The mapping T is called a contraction (strict contraction) if and only if there is a constant $k \in [0,1)$ such that for all $x, y \in D(T)$, we have that $d(Tx, Ty) \leq kd(x,y)$.

- **Definition 2.3.** T is called a Lipschitz map or $L-Lipschitzian$ map if and only if there is a constant $L \geq 0$ such that for all $x, y \in D(T), d(Tx, Ty) \leq Ld(x, y)$.
- **Definition 2.4.** A complete normed linear space is called a Banach space.
- **Definition 2.5.** Let X be a set and let $T: X \to X$ be a function that maps X into itself. (Such a function is often called an operator, a transformation, or a transform on X, and the notation Tx is often used in place of $T(x)$). A fixed point of T is an element $x \in X$ for which $T(x) = x$.
- **Lemma 2.1.** Following Huang and Zhang (2007): Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}_{n\geq 1}$ be a sequence in X. Then $\{x_n\}_{n\geq 1}$ converges to x^* if and only if $d(x_n, x^*) \to 0$ (as $n \to \infty$).
- **Lemma 2.2.** Following Huang and Zhang (2007): Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}_{n\geq 1}$ be a sequence in X. If $\{x_n\}_{n\geq 1}$ converges to x^* and $\{x_n\}_{n\geq 1}$ converges to y^*, then $x^* = y^*$. That is the limit of $\{x_n\}_{n\geq 1}$ is unique.
- **Lemma 2.3.** Following Huang and Zhang (2007): Let (X, d) be a cone metric space, $\{x_n\}_{n\geq 1}$ be a sequence in X. If $\{x_n\}_{n\geq 1}$ converges to x^*, then $\{x_n\}_{n\geq 1}$ is a Cauchy sequence.
- **Lemma 2.4** Following Huang and Zhang (2007): Let (X, d) be a cone metric space, P be a normal cone with normal constant P. Let $\{x_n\}_{n\geq 1}$ be a sequence in X. Then $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ (as $n, m \to \infty$).

Main Result

In this section, we shall prove some fixed point theorem of generalized Lipschitzian map in the setting on cone metric space by defining the sequence $\{x_n\}_{n\geq 1}$ inductively by $x_{n+1} = Tx_n$, $n = 1, 2, \ldots,$ and $x_0 \in X$.

Theorem 3.1. Let (X, d) be a complete metric space and $T: X \to X$ a generalized Lipschitz map with $k < 1$. Define the sequence $\{x_n\}_{n\geq 1}$ inductively by $x_{n+1} = Tx_n$, $n = 1, 2, \ldots,$ $x_0 \in X$ and if x^* is the unique fixed point of T then

i) $x_n \to x^*$ as $n \to \infty$.

ii) $d(x_n, x^*) \leq \dfrac{k^n}{1-k} d(x_1, x_0)$.

Proof

i) Given that (X, d) is a complete metric space. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence. We show that $\{x_{n_j}\}_{j\geq 1}$ a subsequence that converges.

But, $\{x_n\}_{n\geq 1}$ is Cauchy means $\forall \varepsilon > 0, \exists n_\varepsilon \in N : \forall n, m \geq n_\varepsilon, d(x_n, x_m) < \varepsilon$.

$x_{n_j} \to x^*$ as $j \to \infty \Leftrightarrow \varepsilon > 0, \exists j_\varepsilon \in N$ such that $\forall j \geq j_\varepsilon, d(x_{n_j}, x^*) < \varepsilon.$

Set, $N_\varepsilon = \max\{n_\varepsilon, n_{j_\varepsilon}\}$ then $\forall n, m \geq N_\varepsilon, d(x_n, x_m) < \varepsilon, \forall j \geq N_\varepsilon, d(x_{n_j}, x^*) < \varepsilon.$ Therefore, $\forall n, j \geq N_\varepsilon, d(x_n, x^*).$ So that:

$$d(x_n, x^*) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x^*) < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus, $\forall \varepsilon > 0, \exists N_\varepsilon \in N$ such that $\forall n \geq N_\varepsilon, d(x_n, x^*) < 2\varepsilon.$

Hence, $\lim_{n\to\infty} x_n = x^* \Rightarrow x_n \to x^*$ as $n \to \infty.$

ii) Given that $x_0 \in X$ and $x_{n+1} = Tx_{n,} n = 1,2,3,...$

Set $x_1 = Tx_0$

Then, $x_2 = Tx_1$

$x_3 = Tx_2$

$x_4 = Tx_3$

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$x_{n-2} = Tx_{n-3}$

$x_{n-1} = Tx_{n-2}$

$x_n = Tx_{n-1}$

$x_{n+1} = Tx_n$

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We have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1})$$

$$\leq kkd(x_{n-1}, x_{n-2})$$

$$= k^2 d(x_{n-1}, x_{n-2})$$

$$\leq k^2 k d(x_{n-2}, x_{n-3})$$

$$= k^3 d(x_{n-2}, x_{n-3})$$

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$$\leq k^n k d(x_1, x_0).$$

So for $n < m$,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \ldots + d(x_{m-2}, x_{m-1}) + d(x_{m-1}, x_m).$$

$$\leq k^n d(x_1, x_0) + k^{n+1} d(x_1, x_0) + k^{n+2} d(x_1, x_0) + \ldots + k^{m-2} d(x_1, x_0) + k^{m-1} d(x_1, x_0)$$

$$= (k^n + k^{n+1} + k^{n+2} + \ldots + k^{m-2} + k^{m-1}) d(x_1, x_0).$$

$$= k^n d(x_1, x_0)(1 + k^1 + k^2 \ldots + k^{m-n-2} + k^{m-n-1})$$

$$= (1 + k + \ldots + k^{m-n-2} + k^{m-n-1}) k^n d(x_1, x_0)$$

$$\leq (\sum_{i=0}^{\infty} k^i) k^n d(x_1, x_0)$$

$$= \frac{k^n}{1-k} d(x_1, x_0).$$

Therefore, from (i) in Theorem 3.1 above, we have that $x_n \to x^*$ as $n \to \infty$.

Hence, $d(x_n, x^*) \leq \dfrac{k^n}{1-k} d(x_1, x_0).$

Theorem 3.2. Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the generalized Lipschitz condition $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$, where $k \in [0,1)$ is a constant. Then:

i) For any $x \in X$, iterative sequence $\{x_n\}_{n \geq 1}$ inductively defined by $x_{n+1} = Tx_n$, the sequence $\{Tx_n\}_{n \geq 1}$ converges to the fixed point.

ii) T has a unique fixed point in X.

Proof.

i) Choose $x_0 \in X$. Set $x_1 = Tx_0$

Then,

$$x_2 = Tx_1$$

$$x_3 = Tx_2$$

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$$x_{n-2} = Tx_{n-3}$$

$$x_{n-1} = Tx_{n-2}$$

$$x_n = Tx_{n-1}$$

$$x_{n+1} = Tx_n$$

.

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We have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1})$$

$$\leq kkd(x_{n-1}, x_{n-2})$$

$$= k^2 d(x_{n-1}, x_{n-2})$$

$$\leq k^2 kd(x_{n-2}, x_{n-3})$$

$$= k^3 d(x_{n-2}, x_{n-3})$$

.

.

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$$\leq k^n kd(x_1, x_0).$$

So for n>m,

$$d(x_n,x_m)\le d(x_n,x_{n-1})+d(x_{n-1},x_{n-2})+d(x_{n-2},x_{n-3})+\ldots+d(x_{m+2},x_{m+1})+d(x_{m+1},x_m).$$

$$\le k^{n-1}d(x_1,x_0)+k^{n-2}d(x_1,x_0)+\ldots+k^{m+1}d(x_1,x_0)+k^m d(x_1,x_0)$$

$$=(k^{n-1}+k^{n-2}+\ldots+k^{m+1}+k^m)d(x_1,x_0).$$

$$=(k^{n-m-1}+k^{m-n-2}+\ldots+k^1+1)k^m d(x_1,x_0)$$
$$=(1+k+\ldots+k^{m-n-2}+k^{m-n-1})k^m d(x_1,x_0)$$
$$\le(\sum_{i=0}^{\infty}k^i)k^m d(x_1,x_0)$$
$$=\frac{k^m}{1-k}d(x_1,x_0).$$

Since P is normal with a normal constant K, and note that $\|k^m\|\to 0(n\to\infty)$, we have

$$\frac{\|d(x_n,x_m)\|}{\|1-k\|}\le K\|k^m\|.\|d(x_1,x_0)\|.$$

$\Rightarrow d(x_n,x_m)\to 0(n,m\to\infty)$. Hence $\{x_n\}_{n\ge1}$ is a Cauchy sequence. By the completeness of X, there is $x^*\in X\ni x_n\to x^*(n\to\infty)$.

Furthermore, one has

$$d(Tx^*,x^*)\le d(Tx^*,Tx_n)+d(Tx_n,x^*)$$

$$\le kd(x^*,x_n)+d(x_{n+1},x^*)$$

And consequently,

$$\|d(Tx^*,x^*)\|\le K\|(kd(x^*,x_n)+d(x_{n+1},x^*))\|$$

$$\|d(Tx^*,x^*)\|\le K\|(\|k\|.\|d(x_n,x^*)\|+\|d(x^*,x_{n+1})\|)\|\to 0.$$

Hence $\|d(Tx^*,x^*)\|=0\Rightarrow Tx^*=x^*$. So, x^* is a fixed point of T.

ii) Now if $y*$ is another fixed point of $T\ni x^*\ne y^*$, then $d(x^*,y^*)=d(Tx^*,Ty^*)\le kd(x^*,y^*)$.

Hence $\|d(x^*,y^*)\|=0$ and $x^*=y^*$, a contradiction. Therefore the fixed point of T is unique.

Theorem 3.3. Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T:X\to X$ satisfies the generalized Lipschitz condition

$d(Tx,Ty) \le kd(Tx,y) + d(Ty,x) \forall x,y \in X$, where $k \in [0,\frac{1}{2})$ is a constant. Then:

i) For any $x \in X$, iterative sequence $\{x_n\}_{n\ge1}$ inductively defined by $x_{n+1} = Tx_n$, the sequence $\{Tx_n\}_{n\ge1}$ converges to the fixed point.

ii) T has a unique fixed point in X.

Proof

i) Choose $x_0 \in X$. Set $x_1 = Tx_0$

Then,

$$x_2 = Tx_1$$

.

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.

$$x_{n+1} = Tx_n$$

.

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We have

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \le kd(Tx_n,x_{n-1}) + d(Tx_{n-1},x_n)$$

$$\le kd(x_{n+1},x_n) + d(x_n,x_{n-1})$$

So, $d(x_{n+1},x_n) \le \frac{k}{1-k} d(x_n,x_{n-1}) = hd(x_n,x_{n-1}),$

where $h = \frac{k}{1-k}$

For $n > m$,

$$d(x_n,x_m) \le d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2}) + \ldots + d(x_{m+1},x_m)$$

$$\le (h^{n-1} + h^{n-2} + \ldots + h^m) d(x_1,x_0) \le \frac{h^m}{1-h}(d(x_1,x_0)$$

We get $\| d(x_n,x_m) \| \le \frac{h^m}{1-h} K \| d(x_1,x_0) \| \Rightarrow d(x,x) \to 0 (n,m \to \infty).$

Hence, $\{x_n\}_{n\geq 1}$ is a Cauchy Sequence. By the completeness of $X \exists x^* \in X \ni x_n \to x^* (n \to \infty)$.

Since

$$d(Tx^*, x^*) \leq d(Tx_n, Tx^*) + d(Tx_n, x^*)$$

$$\leq k(d(Tx^*, x_n) + d(Tx_n, x^*) + d(x_{n+1}, x^*))$$

$$\leq k(d(Tx^*, x^*) + d(x_n, x^*) + d(x_{n+1}, x^*) + d(x_{n+1}, x^*))$$

$$d(Tx^*, x^*) \leq \frac{1}{1-k}(k(d(x_n, x^*) + d(x_{n+1}, x^*) + d(x_{n+1}, x^*)$$

$$\| d(Tx^*, x^* \| K \frac{1}{1-k}(\| k \| (d(x_n, x^*) \| + \| d(x_{n+1}, x^*) \|) + \| d(x_{n+1}, x^*) \|) \to 0.$$

Hence $d(Tx^*, x^*) = 0$. This implies $Tx^* = x^*$. So x^* is a fixed point of T.

i) Now if y^* is another fixed point of T such that $x^* \neq y^*$, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq k(d(Tx^*, y^*) + d(Ty^*, x^*)) = kd(x^*, y^*) + kd(x^*, y^*) = 2kd(x^*, y^*).$$

Therefore, $d(x^*, y^*) \leq 2kd(x^*, y^*) < 2d(x^*, y^*)$. So that $0 \leq 2d(x^*, y^*)$.

Hence $d(x^*, y^*) = 0 \Rightarrow x^* = y^*$, which is a contradiction.

Therefore the fixed point of T is unique.

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